Stopping near the top of a random walk

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AMS Central Sectional Meeting
Texas Tech University
Lubbock, Tx
April 13, 2014

Talk outline

1. Optimal Stopping problems
2. Stopping near the top of a random walk
    - History/Conjectures
    - New results

Stopping time

Definition

A stopping time with respect to a sequence of random variables $X_1, X_2, ...$ is a random variable τ with values in (1,2,...) and the property that for each t in (1,2,...), the occurrence or non-occurrence of the event $\tau = t$ depends only on the values of $X_1, X_2, ..., X_t$.

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- (iii) From (i) and (ii), if we stop at time k and if $X_1 = x_1, X_2 = x_2, ..., X_k = x_k$, then we receive the reward $Y_k = y_k(x_1, x_2, ..., x_k)$

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When to stop or continue observing variables to maximize the expected payoff or to minimize the expected cost?, That is $E[Y_\tau]$

Finite Horizon Problems

Finite Horizon

Stopping is required after observing $X_1, X_2, ..., X_N$

Backward Induction

We will use backward induction to solve this type of problems.

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- (i) Let $X_1, X_2, ..., X_N$ be independent Bernoulli random variables with parameter p, that is

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1. If $p > \frac{1}{2}$, $\tau = N$ is the unique optimal rule
2. If $p < \frac{1}{2}$, $\tau = 0$ is the unique optimal rule
3. If $p = \frac{1}{2}$, any rule τ such that $P(S_\tau = M_\tau \text{ or } \tau = N) = 1$ is optimal

Stopping near the top of a random walk

Problem

Given $N > 0$, find a stopping time $\tau \leq N$ so as to maximize

$$P(M_N - S_\tau \leq 1).$$

(Win if we stop at one of the two highest values)

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Lemma (Allaart)

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Conclusion

The critical states are $(n, 1)$, for $n = 1, 2, \ldots$.

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Lemma (Allaart)

For each $n \geq 1$*, there exists* $0 < p_n \leq 1$ *such that, in state* $(n, 1)$*, it is optimal to*

- *stop*   *if* $p \leq p_n$;
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Remark

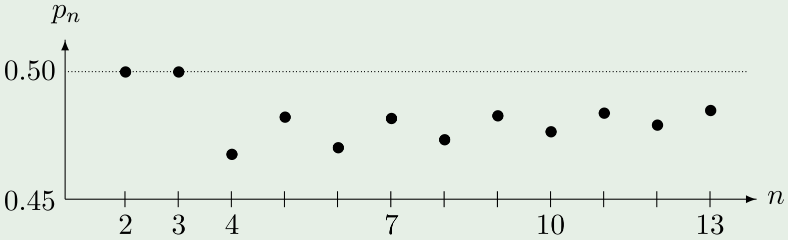
The p_n can be calculated by backward induction.

Table: Critical Values p_n

n	p_n	n	p_n
1	1	11	.48452
2	0.5	12	.47984
3	0.5	13	.48543
4	.46898	14	.48175
5	.48288	15	.48624
6	.47144	16	.48330
7	.48268	17	.48697
8	.47470	18	.48453
9	.48357	19	.48760
10	.47752	20	.48554

Graph and conjectures

Graph of the critical probabilities



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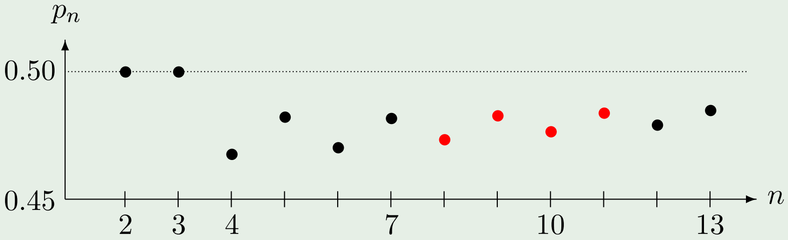

The graph suggests:

1. $p_n < 0.5$ for all $n \geq 4$
2. $\lim_{n\to\infty} p_n = 0.5$
3. $p_{2n-2} < p_{2n} < p_{2n-1} < p_{2n+1}$, for all $n \geq 4$,

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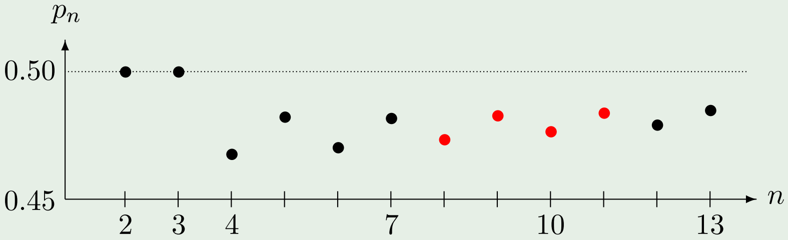

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Conjectures about p_n (Allaart)

i) $\lim_{n\to\infty} p_n = 0.5$.
ii) $p_{2n} \leq p_{2n+2}$ for all $n \geq 2$.

New results

Theorem (J.A.I.)

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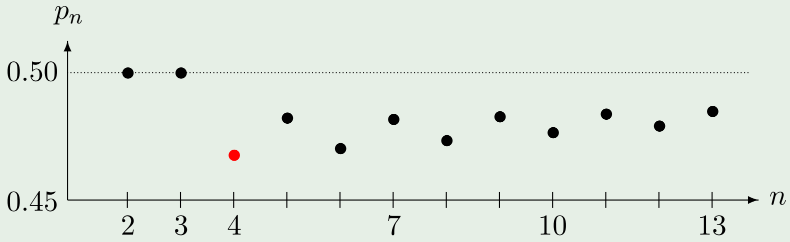

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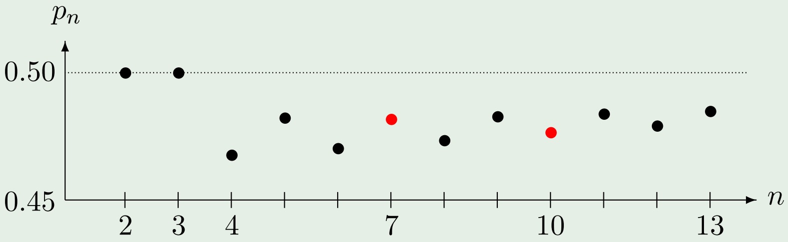

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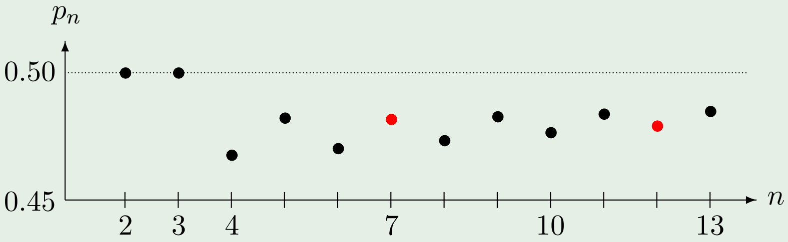

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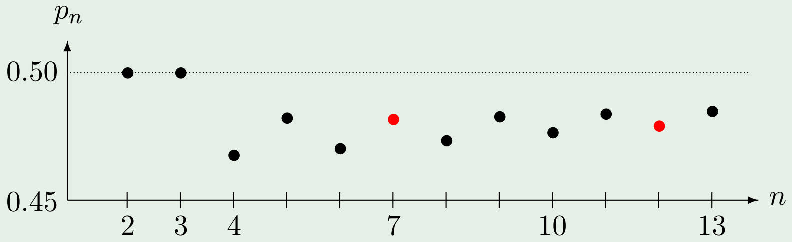

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