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Half the Max

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Outline

1. History: Maximum of a sequence
2. Proportion of the Max
3. Examples
4. Results

History

Maximum of a sequence

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We call this problem: Game Max.

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1. Let F be the distribution function of X_i.
2. An observation X_i is called a candidate if, $X_i = M_i$.
3. For each i, there exists a decision number d_i, such that if X_i is a candidate and $F(x_i) \geq d_i$ then it's optimal to stop at time i.

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Suppose there are n observations and X is a random variable

- Let $V_{n,max}(X,\tau)$ be the probability of win playing the game Max for X using the stopping rule τ.

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- Let $V_{n,max}(X, \tau)$ be the probability of win playing the game Max for X using the stopping rule τ.
- Let τ^* be the optimal stopping rule.
- Let $V^*_{n,max}(X) := \sup_{\tau} V_{n,max}(X, \tau) = V_{n,max}(X, \tau^*)$.

Game Max

For any continuous random variable X

$$v^*_{n,max} := \sup_{\tau} V_{n,max}(X, \tau)$$

Table: (Gilbert and Mosteller)

n	$v^*_{n,max}$	n	$v^*_{n,max}$
1	1.0000	10	.608699
2	.750000	15	.598980
3	.684293	20	.594200
4	.655396	30	.589472
5	.639194	40	.587126
		50	.585725
		∞	.580164

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The win probability does not depend on the distribution of X as long as it is continuous.

Proportion of the Max

Problem

Let $X_1, X_2, ..., X_n$ be iid nonnegative random variables. Given $n > 0$ and $0 < \alpha < 1$, find a stopping time $\tau \leq n$ that maximizes

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Recall the classical prophet inequality $M \leq 2V$, where

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If a gambler wants to achieve a return at least a half of the prophet's return, $P(X_\tau \geq \frac{1}{2} M_n)$, what is his best strategy?

Example 1

Discrete Uniform

$P(X = x) = 1/N$ for $x = 1, 2, ..., N$.

$n = 2$ and $N = 10$

$$\tau^* = \begin{cases} 1, & \lfloor \frac{X_1}{\alpha} \rfloor + \lceil \alpha X_1 \rceil \geq 11 \\ 2, & otherwise. \end{cases}$$

Example 1

Table: Optimal win probabilities when $N = 10$

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$x_1^*(\alpha)$	1	2	3	4	5	5	5	5	6
$V_{2,\alpha max}^*(X)$	1	1	1	0.99	0.98	0.94	0.9	0.86	0.81

Proportion of the Max

Question

When can we win with probability 1?

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Proposition (Allaart, J.I.)

For any $n \geq 1$, *let* X *be a random variable with support on* $[m, M]$ *where* $m > 0$ *and* $M < \infty$. *Consider*

(1) $\alpha^2 \leq \frac{m}{M}$

(2) $P(\frac{m}{\alpha} < X < \alpha M) = 0$

Then $V^*_{n,\alpha max}(X) = 1$ *if and only if* (1) *or* (2) *holds.*

Example 2

Inverse power

$$f(x) = \begin{cases} \frac{a}{x^{a+1}}, & \text{for } x > 1, a > 0 \\ 0, & \text{otherwise.} \end{cases}$$

$n = 2$

$$\tau^* = \begin{cases} 1, & X_1 \geq \left(\alpha^a + \frac{1}{\alpha^a}\right)^{1/a} \\ 2, & otherwise. \end{cases}$$

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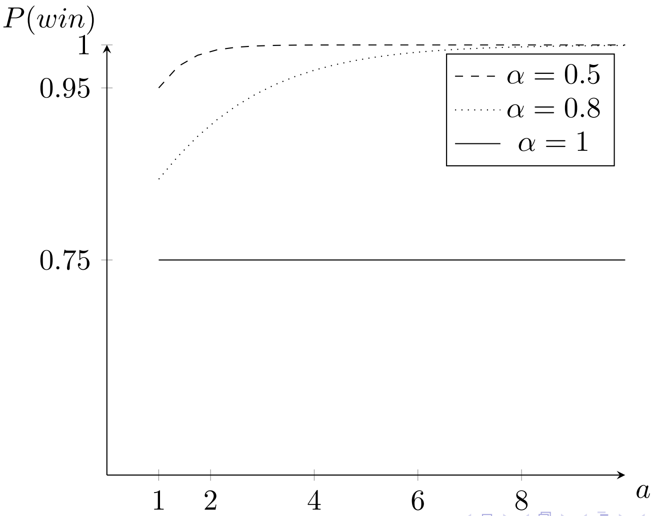
Probability of win

$V^*_{2,\alpha max}(X) = 1 - \frac{\alpha^{3a}}{2(\alpha^{2a}+1)}$.

For fixed α as a goes to infinity the probability of win goes to 1.

Example 2

Figure: Optimal win probabilities for α



Example 2

Inverse power distribution $n = 3$, $\alpha = \frac{1}{2}$

$$\tau^* = \begin{cases} 1, & \frac{7}{2} \leq X_1 \\ 2, & X_1 < \frac{7}{2} \text{ and } X_2 \geq \min\{\frac{5}{2}, g(X_1)\} \\ 3, & otherwise. \end{cases}$$

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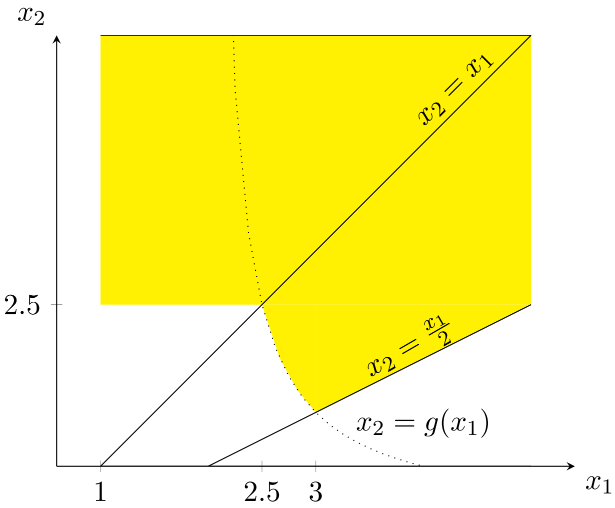
Probability of win

$V^*_{3,hmax}(X) \approx .9846.$

Example 2

Figure: The stopping region for observation 2



Result

Consider the density

$$f_{X^{k,\epsilon}}(x) = \begin{cases} \frac{1}{2\epsilon k} & \text{if } (N_\alpha + 1)^j - \epsilon \leq x \leq (N_\alpha + 1)^j + \epsilon \\ & \quad \text{for } j = 1, ..., k, \\ 0 & \text{otherwise,} \end{cases}$$

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$k \in \mathbb{N}$.

Lemma (Allaart-J.I.)

For every $n \geq 1$, given $\delta > 0$, there exists $k > 1$ such that

$$V_{n,\alpha max}(X^{k,\epsilon}, \tau) \leq V_{n,max}(X^{k,\epsilon}, \tau) + \delta$$

for any stopping rule τ adapted to the filtration $\{\mathcal{F}_i\}$.

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Theorem (Allaart-J.I.)

For each $n \geq 1$, if X is any continuous random variable then $V^*_{n,\alpha max}(X) \geq v^*_{n,max}$ and the bound is sharp.

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Given $\delta > 0$,

$$\begin{aligned} V^*_{n,\alpha max}(X^{k,\epsilon}) &= V_{n,\alpha max}(X^{k,\epsilon}, \tau^*) \\ &\leq V_{n,max}(X^{k,\epsilon}, \tau^*) + \delta \\ &\leq V^*_{n,max}(X^{k,\epsilon}) + \delta \end{aligned}$$

Open problem

Problem

Let $X_1, X_2, ..., X_n$ be independent nonnegative random variables. Given $n > 0$, find a stopping time $\tau \leq n$ that maximizes

$$P(X_\tau = M_n).$$

References

J. Gilbert and F. Mosteller (1966).
Recognizing the maximum of a sequence.
J. Amer. Statist. Assoc. **61**, 35–73.

T. P. Hill and R. P. Kertz (1992).
A survey of prophet inequalities in optimal stopping theory.
Strategies for Sequential Search and Selection in Real Time, Contemporary Mathematics **125**, 191–207.